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Unbiased minimum-variance state and fault estimation for linear systems with unknown input

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Abstract: **This paper extends the existing results on joint input and state estimation to systems with arbitrary fault and unknown inputs. The objective is to derive an optimal filter in the general case where not only fault affect both the systems state and the output, but also the direct feedthrough matrix has arbitrary rank. The paper extends both the results of Bessaoudi and Ben Hmida (2013). [State and fault estimation of linear discrete time systems, (HIS 2013)]. The method is based on the assumption that no prior knowledge about the dynamical evolution of the fault and the disturbance is available. As the fault affects both the state and the output, but the disturbance affects only the state systems. The relationship between the proposed filter and the existing literature results is also addressed. Finally, two numerical examples are given in order to illustrate the proposed method, in particular to solve the estimation of the simultaneous actuator and sensor fault problem and to make a comparison with the existing literature results.**

Keywords: Kalman filtering, Recursive state estimation , fault estimation, minimum-variance estimation.

I. Introduction

In the past few years, the problem of filtering in the presence of unknown inputs has attracted big attention, due to its applications in environment. The unknown input filtering problem has been treated in the literature by different approaches. The first approach assumes that the model for dynamical evolution of the unknown inputs is available. When the properties of the unknown input are known, the augmented state Kalman filter (ASKF) is a solution. To reduce computation costs of the ASKF, Friedland [1] proposed the two stage Kalman filter where the estimation of the state and unknown input are decoupled. The second approach treats the case when we not have a prior knowledge about the dynamical evolution for unknown input. Kitanidis [2] was the first to solve the problem using the linear unbiased minimum-variance. An extend Kitanidis filter using a paramaterizing technique to obtain an optimal filter (OEF) have been proposed by Darouach et al [3]. Hseih [4] has been developed a robust-two stage Kalman filter (RTSKF) equivalent to Kitanidis filter. An (OMVF) reported by C.S Hsieh [5] has been used in order to developed an optimal minimum variance filter (OMVF) to solve degradation problems encountered in (OEF). Gillijns and De Moor [6] has treated the problem of estimating the state in the presence of unknown inputs which affect the systems model. They have been developed a recursive filter which is optimal in the sense of minimum-variance. This filter has been extended by the same authors for joint input and state estimation to linear discrete-time systems with direct feedthrough where the state and the unknown input estimation are interconnected. This filter is called recursive three step filter (RTSF) [7] and is limited to direct feedthrough matrix with full rank. Cheng et al, [8] proposed a recursive optimal filter with global optimality in the sense of unbiased minimum-variance. This filter is limited to estimate the state. The case of an arbitrary rank has been proposed by Hsieh in [9] the designed optimal filter Known as ERTSF (Extend RTSF). Recently, another technique using a least square method have been proposed by Bessaoudi et al, [10] to estimate the state and unknown input. The Fault Detection and Isolation (FDI) problem for linear systems with unknown disturbances is generally studied, see e.g. Nikoukhah [11], Keller [12], Chen and Patton [13, 14], Ben Hmida et al, [15]. According to [11], a robust fault detection and isolation in continuous-time is developed using the error innovation technique to generate an unbiased white residual signals. The fault is diagnosed by a statistical testing. A new method is developed in order to detect and isolate multiple faults appearing simultaneousely or sequentially in linear time-invariant stochastic discrete-time systems with unknown inputs [12]. Their methods consist of generating directional residuals using an isolation filter. In [13] the optimal filtering and robust fault diagnosis problem has been studied for stochastic systems with unknown disturbances. The output estimation error with disturbance decoupling is used as a residual signal. After that, a statistical testing procedure is applied to examine the residual and to diagnose faults. Netherless, the simultaneous actuator and sensor fault and state problem is not treated in [13, 14]. Recently, [16]

present a new optimal recursive filter for state and fault estimation of linear stochastic systems with unknown disturbances. This method is based on the assumption that no prior knowledge about the dynamical evolution of the unknown disturbances is available. The filter has two advantages: it considers an arbitrary direct feedthrough matrix of the fault and it permits a multiple faults estimations. Ben Hmida et al [18] present a (OThSKF), this filter is obtained after decoupling the covariance matrices of the augmented state Kalman filter using a three-stage U-V transformation. A new robust filter based on recursive least square estimation for linear stochastic systems with unknown disturbances are proposed in [10], the novel elements of this algorithm are an easily simple implementable, square root method is used to solve the numerical problems affecting the unknown input filter algorithm and related information filter and smoothing algorithms. The main objective of this paper is to develop a new filter that can solve the problem of simultaneously estimating the state and the fault in the presence of the unknown input.
The remainder of this paper is organized as follows. Section 2 states the problem of interest and some preliminary. In Section 3, the design of filter is developed. Finally, an illustrative example of the proposed approach techniques is presented.

II. Problem and Preliminaries

A. Problem formulation

The problem consists of designing a filter that gives a robust state and fault estimation for linear time-varying stochastic systems in the presence of unknown inputs. We consider that the linear stochastic time-varying discrete stochastic systems with unknown disturbances and additive faults are described by the following form:

$$x_{k+1} = A_k x_k + B_k u_k + F_k^x f_k + G_k d_k + w_k \quad (1)$$
$$y_k = C_k x_k + D_k u_k + F_k^y f_k + v_k \quad (2)$$

where $x_k \in \Re^n$ is the state vector, $y_k \in \Re^m$ is the observation vector, $u_k \in \Re^r$ is the known input vector, $f_k \in \Re^p$ is the additive fault vector and $d_k \in \Re^q$ is the unknown disturbances vector. The matrices A_k, B_k, C_k, D_k F_k^x, G_k and F_k^y are known and have appropriate dimensions. f_k presents the vector of an additive fault that can be occur in the systems. d_k the unknown input vector, can be present an unknown perturbation : for example a parametric uncertainty.
The following assumptions are necessary for the development of the filter.
Assumptions:

1. The noises w_k and v_k are zero-mean white noise sequence with the following covariances :

$$\varepsilon\left[w_k w_l^T\right] = Q_k \delta_{kl} \text{ and } \varepsilon\left[v_k v_l^T\right] = R_k \delta_{kl}.$$

   where ^{T} denotes transpose and δ_{kl} denotes the Kronecker delta function.

2. The process noise w_k and the measurement noise v_k are uncorrelated.

$$\varepsilon\left[w_k v_l^T\right] = \varepsilon\left[v_k w_l^T\right] = 0$$

3. The initial state is a Gaussian random variable and is uncorrelated with the white noise processes w_k and v_k : $\varepsilon(x_0) = \hat{x}_0$ and $P_0^x = \varepsilon\left(\left(x_0 - \hat{x}_0\right)\left(x_0 - \hat{x}_0\right)^T\right)$ where $\mathcal{E}[.]$ denotes the expectation operator.

4. Conditions on matrices ranks: $0 < rank\left(F_k^y\right) \leq p$ and $rank\left[C_k G_{k-1}\right] = rank\left[G_{k-1}\right] = q$

5. The pair $(C_k,\, A_k)$ is observable.

The objective of this paper is extend the results of [17] in order to derive a new recursive optimal filter structure to obtain a better fault and state estimation when $0 < rank\ (F_k^y) \leq p$ in spite of the presence of the unknown disturbance d_k.

B. Preliminary material

We first carry out a transformation of the system to decouple the output equation into two components, one with a full rank direct feedthrough matrix and the other without direct feedthrough. In this form, the filter can be designed leveraging existing approaches for both cases [7, 8].
Let $r_k = rank\left(F_k^y\right) < p$. Then the singular value decomposition of the matrix F_k^y is given by:

$$F_k^y = \begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,k}^T \\ V_{2,k}^T \end{bmatrix} \quad (3)$$

where we have $\Sigma_k \in \Re^{r_k \times r_k}$, $U_{1,k} \in \Re^{m \times r_k}$, $U_{2,k} \in \Re^{m \times (m-r_k)}$, $V_{1,k} \in \Re^{p \times r_k}$ and $V_{2,k} \in \Re^{p \times (p-r_k)}$.
$\begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix}$ and $\begin{bmatrix} V_{1,k} & V_{2,k} \end{bmatrix}$ are unitary matrices. There existe a transformation matrix of the from $T_k = \begin{bmatrix} T_{1,k}^T & T_{2,k}^T \end{bmatrix}^T$ such that the systems (1) and (2) written as [8]:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + F_{1,k}^x f_{1,k} + F_{2,k}^x f_{2,k} \\ &\quad + G_k d_k + w_k && (4) \\ z_{1,k} &= C_{1,k} x_k + D_{1,k} u_k + \Sigma_k f_{1,k} + v_{1,k} && (5) \\ z_{2,k} &= C_{2,k} x_k + D_{2,k} u_k + v_{2,k} && (6) \end{aligned}$$

where

$$\begin{aligned} v_{1,k} &= T_{1,k} v_k \\ v_{2,k} &= T_{2,k} v_k \\ z_{1,k} &= T_{1,k} y_k \in \Re^{r_k} \\ z_{2,k} &= T_{2,k} y_k \in \Re^{(m-r_k)} \\ F_{1,k}^x &= F_k^x V_{1,k} \\ F_{2,k}^x &= F_k^x V_{2,k} \\ C_{1,k} &= T_{1,k} C_k \\ C_{2,k} &= T_{2,k} C_k \\ D_{1,k} &= T_{2,k} D_k \\ D_{2,k} &= T_{2,k} D_k \end{aligned} \quad (7)$$

with assumption that:

$$rank(C_{2,k}F^x_{2,k-1}) = rank(F^x_{2,k-1}) \quad (8)$$

$$rank(C_{2,k}G_{k-1}) = rank(G_{k-1}) \quad (9)$$

Since $[V_{1,k}\ V_{2,k}]$ is a unitary matrix, it follows that the fault must be reconstructed through its two components $f_{1,k}$ and $f_{2,k}$ according to:

$$f_k = V_{1,k}f_{1,k} + V_{2,k}f_{2,k} \quad (10)$$

The transformation matrix T_k is given by:

$$T_k = \begin{bmatrix} I_{r_k} & -U^T_{1,k}R_kU_{2,k}\left(U^T_{2,k}R_kU_{2,k}\right)^{-1} \\ 0_{(m-r_k)\times r_k} & I_{(m-r_k)} \end{bmatrix} \times \begin{bmatrix} U^T_{1,k} \\ U^T_{1,k} \end{bmatrix} \quad (11)$$

Cheng et al. [8] define $R_{1,k}$ and $R_{,k}$ as the variance of $v_{1,k}$ and $v_{2,k}$, respectively, and $R_{12}(k,i)$ as their covariance. Then it follows that:

$$\begin{aligned} R_{1,k} &= \mathcal{E}[v_{1,k}v^T_{1,k}] \\ &= U^T_{1,k}R_kU_{1,k} - U^T_{1,k}R_kU_{2,k} \\ &\quad \times \left(U^T_{2,k}R_kU_{2,k}\right)^{-1}U^T_{2,k}R_kU_{1,k} \\ R_{2,k} &= \mathcal{E}[v_{2,k}v^T_{2,k}] = U^T_{2,k}R_kU_{2,k} \\ R_{12}(k,k) &= \mathcal{E}[v_{1,k}v^T_{2,k}] = 0 \\ R_{12}(k,i) &= \mathcal{E}[v_{1,k}v^T_{2,i}] = 0 \quad for k \neq i \end{aligned}$$

Moreover, Cheng et al. [8] show the following relations:

- $cov\,[v_{1,k}, w_i] = 0$ and $cov\,[v_{2,k}, w_i] = 0$ for $k \neq i$.
- $cov\,[v_{1,k}, x_0] = 0$ and $cov\,[v_{2,k}, x_0] = 0$.

Under the system equations (4)-(6), we note that we can estimate the second component of the fault not at the step k but at step $k-1$, because $f_{2,k-1}$ will be estimated from the state. The proposed state and fault filter has the following structure:

$$\begin{aligned} \hat{x}_{k/k-1} &= A_{k-1}\hat{x}_{k-1/k-1} + B_{k-1}u_{k-1} + \\ &\quad F^x_{1,k-1}\hat{f}_{1,k-1}, && (12) \\ \hat{f}_{2,k-1} &= M^{f_2}_k\left(z_{2,k} - D_{2,k}u_k - C_{2,k}\hat{x}_{k/k-1}\right) && (13) \\ \hat{f}_{1,k} &= M^{f_1}_k\left(z_{1,k} - D_{1,k}u_k - C_{1,k}\hat{x}^*_{k/k-1}\right) && (14) \\ \hat{f}_k &= V_{1,k}\hat{f}_{1,k} + V_{2,k}\hat{f}_{2,k-1} && (15) \\ \hat{x}^*_{k/k-1} &= \hat{x}_{k/k-1} + F^x_{2,k-1}\hat{f}_{2,k-1} + \\ &\quad K^{x*}_k\left(z_{2,k} - C_{2,k}\hat{x}_{k/k-1}\right) && (16) \\ \hat{x}_{k/k} &= \hat{x}^*_{k/k-1} + \\ &\quad K^x_k\left(z_{2,k} - D_{2,k}u_k - C_{2,k}\hat{x}^*_{k/k-1}\right) && (17) \end{aligned}$$

where the gain matrices $M^{f_1}_k$,$M^{f_2}_k$, K^{x*}_k and K^x_k are determined to satisfy the following criteria:

- ***Unbiasedness***: the estimator must satisfy

$$\begin{aligned} \mathcal{E}[\tilde{f}_k] &= \mathcal{E}[f_k - \hat{f}_k] = 0 && (18) \\ \mathcal{E}[\tilde{x}_{k/k}] &= \mathcal{E}[x_k - \hat{x}_{k/k}] = 0. && (19) \end{aligned}$$

- ***Minimum-variance*** : the estimator is determined such that:

- The mean square errors $\mathcal{E}[(\tilde{f}_k)^T\tilde{f}_k]$ is minimized under the constraint (18),
- The $trace\left\{P^x_{k/k} = \mathcal{E}[\tilde{x}_{k/k}(\tilde{x}_{k/k})^T]\right\}$ is minimized under the constraints (18) and (19).

III. Filter design

A. Time update

First, we consider the time update. Let the $\hat{x}_{k-1/k-1}$ and $\hat{f}_{k/k}$ denote the optimal unbiased estimates of x_{k-1} and f_{k-1} given measurement up to time $k-1$, then the time update is given by equation (12) and (15).
With $\tilde{x}_{k/k} = x_k - \hat{x}_{k/k}$ and $\tilde{f}_{k/k} = f_k - \hat{f}_{k-1}$. Consequently, the covariance matrix $\bar{P}^x_{k/k-1}$ has the following form:

$$\begin{aligned} \bar{P}^x_{k/k-1} &= \varepsilon\left[\bar{x}_{k/k-1}\bar{x}^T_{k/k-1}\right] \\ &= \begin{bmatrix} A_{k-1} & F^x_{1,k-1} \end{bmatrix} \begin{bmatrix} P^x_{k-1/k-1} & P^{xf_1}_{k-1} \\ \left(P^{xf_1}_{k-1}\right)^T & P^{f_1}_{k-1} \end{bmatrix} \begin{bmatrix} A^T_{k-1} \\ F^{xT}_{1,k} \end{bmatrix} + Q_{k-1} \end{aligned}$$

with $P^x_{k/k-1} = \varepsilon\left[\tilde{x}_{k/k-1}\tilde{x}^T_{k/k-1}\right]$, $P^{f_1}_{k/k-1} = \varepsilon\left[\tilde{f}_{1,k/k-1}\tilde{f}^T_{1,k/k-1}\right]$ and $P^{xf_1}_{k-1} = \varepsilon\left[\tilde{x}_{k-1/k-1}\tilde{f}^T_{1,k-1}\right]$. Expression for this covariance matrices will be derived in the next sections.

B. Fault estimation

In this section, we consider the estimation of the fault. The gain matrices $M^{f_1}_k$ and $M^{f_2}_k$ will be determined so that the filter yields robust estimates of f_k and x_k in spite of the presence of the unknown disturbances d_k. Next, the unbiased minimum-variance fault and state estimation will be demonstrated. The errors estimation of $f_{1,k}$ and $f_{2,k-1}$ are given by:

$$\begin{aligned} \tilde{f}_{1,k} &= f_{1,k} - \hat{f}_{1,k} \\ &= \left(I - M^{f_1}_k\Sigma_k\right)f_{1,k} - M^{f_1}_k\tilde{y}_{1,k} \end{aligned} \quad (20)$$

$$\begin{aligned} \tilde{f}_{2,k-1} &= f_{2,k-1} - \hat{f}_{2,k-1} \\ &= \left(I - M^{f_2}_kC_{2,k}F^x_{2,k-1}\right)f_{2,k-1} \\ &\quad -M^{f_2}_kC_{2,k}G_{k-1}d_{k-1} - M^{f_2}_k\tilde{y}_{2,k} \end{aligned} \quad (21)$$

where

$$\begin{aligned} \tilde{y}_{1,k} &= C_{1,k}\tilde{x}^*_{k/k-1} + v_{1,k}, && (22) \\ \tilde{x}^*_{k/k-1} &= x_k - \hat{x}^*_{k/k-1}, && (23) \\ \tilde{y}_{2,k} &= C_{2,k}\bar{x}_{k/k-1} + v_{2,k} && (24) \\ \bar{x}_{k/k-1} &= A_k\tilde{x}_{k-1/k-1} + F^x_{1,k-1}\tilde{f}_{1,k-1} + w_{k-1} && (25) \end{aligned}$$

Referring to (20) and (21), the estimator $\hat{f}_k$ given by (15) is unbiased if and only if $M^{f_1}_k$ and $M^{f_2}_k$ satisfy the following constraints :

$$M^{f_1}_k\Sigma_k = I_{r_k} \quad (26)$$

$$M^{f_2}_kE_k = \Gamma_k \quad (27)$$

where $E_k = \begin{bmatrix} C_{2,k}F^x_{2,k-1} & C_{2,k}G_{k-1} \end{bmatrix}$ and $\Gamma_k = \begin{bmatrix} I_{p-r_k} & 0_{(p-r)q} \end{bmatrix}$.
Let $\tilde{x}^*_{k/k-1}$, $\tilde{x}_{k-1/k-1}$ and $\tilde{f}_{1,k-1}$ be unbiased. The covariance matrices of $\tilde{y}_{1,k}$ and $\tilde{y}_{2,k}$ are defined respectively by :

$$\begin{aligned}\tilde{R}_{1,k} &= \varepsilon\left[\tilde{y}_{1,k}\tilde{y}^T_{1,k}\right]\\ &= C_{1,k}P^{x*}_{k/k-1}C^T_{1,k} + R_{1,k}\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde{R}_{2,k} &= \varepsilon\left[\tilde{y}_{2,k}\tilde{y}^T_{2,k}\right]\\ &= C_{2,k}\bar{P}^x_{k/k-1}C^T_{2,k} + R_{2,k}\end{aligned} \tag{29}$$

where $P^{x*}_{k/k-1} = \varepsilon\left[\tilde{x}^*_{k/k-1}\tilde{x}^{*T}_{k/k-1}\right]$ and $\bar{P}^x_{k/k-1} = \varepsilon\left[\bar{x}^*_{k/k-1}\bar{x}^{*T}_{k/k-1}\right]$.
Since the errors estimations $\tilde{y}_{1,k}$ and $\tilde{y}_{2,k}$ has unit variances the least-squares (LS) solutions do not have a minimum-variance. For that $f_{1,k}$ and $f_{2,k}$ can be obtained by weighted least-squares estimation [20] with two weighting matrices $\tilde{R}^{-1}_{1,k}$ and $\tilde{R}^{-1}_{,k}$. Then, to have unbiased fault estimates, the matrices gain $M^{f_1}_k$ and $M^{f_2}_k$ are obtained as follows :

$$M^{f_1}_k = \left(\Sigma^T_k\tilde{R}^{-1}_{1,k}\Sigma_k\right)^{-1}\Sigma^T_k\tilde{R}^{-1}_{1,k} \tag{30}$$

$$M^{f_2}_k = \Gamma_k E^*_k \tag{31}$$

where

$$E^*_k = \left(E^T_k\tilde{R}^{-1}_{2,k}E_k\right)^+ E^T_k\tilde{R}^{-1}_{2,k} \tag{32}$$

is the generalized inverse of the matrix E_k.
The variances of the WLS solutions (20) and (21) are respectively given by:

$$P^{f_1}_k = \varepsilon\left[\tilde{f}_{1,k}\tilde{f}^T_{1,k}\right] = \left(\Sigma^T_k\tilde{R}^{-1}_{1,k}\Sigma_k\right)^{-1}, \tag{33}$$

$$P^{f_2}_k = \varepsilon\left[\tilde{f}_{2,k-1}\tilde{f}^T_{2,k-1}\right] = M^{f_2}_k\tilde{R}^{-1}_{2,k}\left(M^{f_2}_k\right)^T \tag{34}$$

Referring to equations (10), (15), (20) and (21), the fault error estimate $\tilde{f}_k$ has the following form:

$$\tilde{f}_k = \begin{bmatrix} V_{1,k} & V_{2,k} \end{bmatrix}\begin{bmatrix} \tilde{f}_{1,k} \\ \tilde{f}_{2,k} \end{bmatrix} \tag{35}$$

Using (35), the covariance matrix P^f_k is given by

$$P^f_k = \begin{bmatrix} V_{1,k} & V_{2,k} \end{bmatrix}\begin{bmatrix} P^{f_1}_k & P^{12}_k \\ P^{f_{21}}_k & P^{f_2}_K \end{bmatrix}\begin{bmatrix} V^T_{1,k} \\ V^T_{2,k} \end{bmatrix} \tag{36}$$

where

$$\begin{aligned}P^{f_{12}}_k &= \left(P^{f_{21}}_k\right)^T = \varepsilon\left[\tilde{f}_{1,k}\tilde{f}^T_{2,k-1}\right]\\ &= M^{f_1}_k C_{1,k}\left[P^x_{k/k-1}C^T_{2,k} + \tilde{S}_k\right]\left(M^{f_2}_k\right)^T\end{aligned} \tag{37}$$

with $\tilde{S}_k = \varepsilon\left[\tilde{x}^*_{k/k-1}v^T_{2,k}\right] = -\left(F^x_{k-1}M^{f_2}_k + K^{x*}_k\right)\tilde{R}_{2,k}$.

C. State estimation

In this subsection, we consider the estimation of the state. The gain matrices K^{x*}_k and K^x_k will be determined so that the filter yields robust estimates state x_k in spite of the presence of the fault f_k and the unknown disturbances d_k . Referring to equations (1) and (16) the state estimations error $\tilde{x}^*_{k/k-1}$ is defined as

$$\begin{aligned}&\tilde{x}^*_{k/k-1} = \bar{x}_{k/k-1} - \left(K^{x*}_k + F^x_{2,k-1}M^{f_2}_k\right)\tilde{y}_{2,k}\\ &+ \left(G_{k-1} - K^{x*}_k C_{2,k}G_{k-1}\right)d_{k-1}\\ &-K^{x*}_k C_{2,k}F^x_{2,k-1}f_{2,k-1}\end{aligned} \tag{38}$$

The estimator $\hat{x}^*_{k/k-1}$ is unbiased if K^{x*}_k satisfies the following constraint to eliminate the terms $f_{2,k-1}$ and d_{k-1} from the error estimate (38).

$$K^{x*}_k E_k = \Gamma^*_k \tag{39}$$

where $\Gamma^*_k = \begin{bmatrix} 0_{n\times(p-r_k)} & G_{k-1} \end{bmatrix}$.
Lemma: The necessary and sufficient condition so that the estimators (4) and (5) are unbiased as matrix E_k is full column rank, i. e.

$$rank(E_k) = rank(F^x_{2,k-1}) + rank(G^x_{k-1}) \tag{40}$$

Proof: The equations (27) and (39) can be written

$$\begin{bmatrix} M^{f_2}_k \\ K^{x*}_k \end{bmatrix} E_k = \begin{bmatrix} \Gamma_k \\ \Gamma^*_k \end{bmatrix}. \tag{41}$$

A necessary and sufficient condition for the existence of the solution to (41) is

$$rank\begin{bmatrix} \Gamma_k \\ \Gamma^*_k \\ E_k \end{bmatrix} = E_k \tag{42}$$

We expand (42) and obtain

$$rank\begin{bmatrix} \Gamma_k \\ \Gamma^*_k \\ E_k \end{bmatrix} = rank\begin{bmatrix} I_{p-r_k} & 0_{(p-r_k)\times q} \\ 0_{n\times(p-r_k)} & G_{k-1} \\ C_{2,k}F^x_{2,k-1} & C_{2,k}G_{k-1} \end{bmatrix}$$

$$= rank\begin{bmatrix} C_{2,k}F^x_{2,k-1} & C_{2,k}G_{k-1} \end{bmatrix}$$

$$= rank\left(C_{2,k}F^x_{2,k-1}\right) + rank\left(C_{2,k}G_{k-1}\right)$$

Finally, referring to the equations (8) and (9) , we will have

$$rank(E_k) = rank(F^x_{2,k-1}) + rank(G^x_{k-1})$$

However, this can be easily justified by considering that the fault and the unknown disturbances have independent influences.
Referring to (38) and (39), the covariance matrix $P^{x*}_{k/k-1}$ has the following form:

$$\begin{aligned}P^{x*}_{k/k-1} &= \mathcal{E}\left[\tilde{x}^*_{k/k-1}(\tilde{x}^*_{k/k-1})^T\right]\\ &= (I_n - F^x_{2,k-1}M^{f_2}_k C_{2,k} - K^{x*}_k C_{2,k})\overline{P}^x_{k/k-1}\\ &\times(I_n - F^x_{2,k-1}M^{f_2}_k C_{2,k} - K^{x*}_k C_{2,k})^T\\ &+(F^x_{2,k-1}M^{f_2}_k + K^{x*}_k)R_{2,k}\\ &\times(F^x_{2,k-1}M^{f_2}_k + K^{x*}_k)^T.\end{aligned} \tag{43}$$

The gain matrix is determined by minimizing the trace of the covariance matrix (43) such that (39) is satisfied. Using the Kitanidis method we obtain

$$\begin{bmatrix} \tilde{R}_{2,k} & -E_k \\ E_k^T & 0 \end{bmatrix} \begin{bmatrix} K_k^{x*T} \\ \Lambda_k^{*T} \end{bmatrix} = \begin{bmatrix} \tilde{R}_{2,k} M_k^{f_2 T} F_{2,k-1}^{xT} + C_{2,k} \bar{P}_{k/k-1}^{x} \\ \Gamma_k^* \end{bmatrix} \tag{44}$$

where Λ_k^* is the matrix of lagrange multipliers.
Equations (41) will have a unique solution. Accordingly, the gain matrix K_k^{x*} is given by

$$K_k^{x*} = \left(\bar{P}_{k/k-1}^{x} C_{2,k}^T - F_{2,k-1}^{x} M_k^{f_2} \tilde{R}_{2,k} \right) \times \tilde{R}_{2,k}^{-1} \left(I - E_k E_k^* \right) + \Gamma_k^* E_k^* \tag{45}$$

Using (1) and (17), the state estimation error $\tilde{x}_{k/k}$ has the following form:

$$\tilde{x}_{k/k} = \left(I - K_k^x C_{2,k} \right) \tilde{x}_{k/k-1}^* - K_k^x v_{2,k} \tag{46}$$

Considering (44), the covariance matrix is determined as follows:

$$P_{k/k}^x = \varepsilon \left[\tilde{x}_{k/k} \tilde{x}_{k/k}^T \right] = P_{k/k-1}^{x*} + K_k^x R_k^* K_k^{xT} - V_k^* K_k^{xT} - K_k^x V_k^{*T} \tag{47}$$

where

$$\begin{aligned} R_k^* &= C_{2,k} P_{k/k-1}^{x*} C_{2,k}^T + R_{2,k} + C_{2,k} S_k^* + \left(C_{2,k} S_k^* \right)^T \\ V_k^* &= P_{k/k-1}^{x*} C_{2,k}^T + S_k^* \\ S_k^* &= \varepsilon \left[\tilde{x}_{k/k-1} v_{2,k}^T \right] = -\left(F_{2,k-1}^{x} M_k^{f_2} + K_k^{x*} \right) R_{2,k} \end{aligned}$$

In order to obtain a minimum-variance estimate, we have to minimize the trace of (46). Thus, the gain matrix K_k^x is given by:

$$K_k^x = \left(P_{k/k-1}^{x*} C_{2,k}^T + S_k^* \right) \beta_k^T \left(\beta_k R_k^* \beta_k^T \right) \beta_k \tag{48}$$

Where β_k is an arbitrary matrix which has to be chosen such that $\beta_k R_k^* \beta_k^T$ has full rank.
The matrix $P_k^{xf_1}$ is calculated by using (20) and (46), then we obtain:

$$\begin{aligned} P_k^{xf_1} &= \varepsilon \left[\tilde{x}_{k/k} f_{1,k}^T \right] \\ &= \left(K_k^x (V_k^*)^T - P_{k/k-1}^{x*} \right) \left(M_k^{f_1} C_{1,k} \right)^T \end{aligned} \tag{49}$$

Remark:

1. If $G_k = B_k = D_k = 0$ and $0 < rank\left(F_k^y\right) \le p$ the obtained filter is equivalent to ERTSF developed by [9].
2. If $G_k = B_k = D_k = 0$ and $rank\left(F_k^y\right) = p$ the obtained filter is equivalent to RTSF developed by [7].
3. In the case where $F_k^x = B_k = D_k = 0$ and $F_k^y = 0$ the filter [2] is obtained.
4. In the case where $F_k^x = 0$,$F_k^y = 0$ and $G_k = 0$ we obtain the standard Kalman filter.

IV. Applications

In this section, we propose the use of the resulting filter to solve the estimation of simultaneous actuator and sensor faults problem and to make a comparison with the existing literature results in particular the ones of [17].

A. Illustrative example

We consider the same numerical example used in [**?**, **?**]. The linearized model of a simplified longitudinal flight control system is the following:

$$\begin{aligned} x_{k+1} &= (A_k + \Delta A_k) x_k + (B_k + \Delta B_k) u_k \\ &\quad + F_k^a f_k^a + w_k \\ y_k &= C_k x_k + F_k^s f_k^s + v_k \end{aligned}$$

where the state variables are the pitch angle δ_z, the pitch rate ω_z and the normal velocity η_y. The control input u_k is the elevator control signal. F_k^a and F_k^s are the matrices distribution of the actuator fault f_k^a and sensor fault f_k^s.

The presented system equations can be rewritten as follow:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + F_k^x f_k + G_k d_k + w_k \\ y_k &= C_k x_k + F_k^y f_k + v_k \end{aligned}$$

where F_k^x and F_k^y are the matrices injection of the faults vector in the state and measurement equations.

$$\begin{aligned} F_k^x &= [F_k^a \quad 0] \\ F_k^y &= [0 \quad F_k^s] \end{aligned}$$

The term $G_k d_k$ represents the parameter perturbation in matrices A_k and B_k:

$$G_k d_k = \Delta A_k x_k + \Delta B_k u_k \tag{50}$$

The system parameter matrices are:

$$A_k = \begin{bmatrix} 0.9944 & 0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix}, \; x_k = \begin{bmatrix} \delta_z \\ w_z \\ \eta_z \end{bmatrix}$$

$$B_k = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, C_k = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.5 & 0.1 & 0.25 \\ 0.1 & 0.2 & 0.3 \end{bmatrix},$$

$Q_k = diag\left\{0.1^2, 0.1^2, 0.01^2\right\}$, $R_k = 0.1^2(I_{4\times 4})$
We inject simultaneously two faults in the system,

$$\begin{bmatrix} f_k^a \\ f_k^s \end{bmatrix} = \begin{bmatrix} 4u_s(k-20) - 4u_s(k-60) \\ -2u_s(k-30) + 2u_s(k-65) \end{bmatrix}$$

where $u_s(k)$ is the unit-step function. The first fault f_k^a occurs in the actuator between 20 and 60 and the second fault f_k^s occurs in the sensor for δ_z between 30 and 65. Thus, we consider the cases of single and multiples faults. The unknown disturbance is given by:

$$G_k d_k = \Delta A_k x_k + \Delta B_k u_k \tag{51}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{G_k} \underbrace{\left\{ \begin{bmatrix} \Delta a_{11} & \Delta a_{12} & \Delta a_{13} \\ \Delta a_{21} & \Delta a_{22} & \Delta a_{23} \end{bmatrix} x_k + \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} u_k \right\}}_{d_k} \tag{52}$$

Table 1: : Evaluation of the RMSE values

$\tilde{x}_{1,k}$	$\tilde{x}_{2,k}$	$\tilde{x}_{3,k}$	$\tilde{f}_{1,k}$	$\tilde{f}_{2,k}$	$trace\, P_k^x$	$trace\, P_k^f$
0.7789	0.1941	0.2642	0.0804	0.2646	0.7772	0.0848

where Δa_{ij} and Δb_i $(i = 1, j = 1, 2, 3)$ are perturbations in aerodynamic and control coefficients.
In the simulation, the aerodynamic coefficients are perturbed by $\pm 50\%$, i.e $\Delta a_{ij} = -0.5 a_{ij}$ and $\Delta b_i = -0.5 b_i$.
We set $P_0 = 0.1^2 eye(3)$, $x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $u_k = 10$.
In the previous table (Table 1), the Root Mean Square Errors are given along with the traces of their steady-fault and state estimation error covariances.

It can be seen that the proposed filter produces better estimates of the state and faults. We primarily focus on simultaneous estimation of actuator and sensor faults in spite of the presence of unknown disturbances.

B. Comparative Study

In this section, we will apply the proposed filter to treat two different cases. The parameters of the systems (1) are given by:

$$A_k = \begin{bmatrix} a_k & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.1 & 0.25 \end{bmatrix}, D_k = \begin{bmatrix} 1 \\ 0.5 \\ 1.5 \end{bmatrix}$$

$$B_k = \begin{bmatrix} 2 \\ -1.5 \\ 0.5 \end{bmatrix}, F_k^x = \begin{bmatrix} 0.5 & 0.7 \\ 1.5 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}, G_k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_k = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, Q_k = 0.1 eye(3),$$

$R_k = 0.1^2\, eye\,(3)$, $a_k = 0.4 + 0.3\sin(0.2k)$, $x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix}$

The initial value of the state is $x_0 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\bar{P}_0^x = eye(3)$.
The fault and the unknown input are given by:

$$\begin{bmatrix} f_k^1 \\ f_k^2 \end{bmatrix} = \begin{bmatrix} 4u_s\,(k-25) + 4u_s\,(k-70) \\ 4u_s\,(k-30) + 4u_s\,(k-65) \end{bmatrix}$$

$d_k = 4u_s\,(k-15) + 4u_s\,(k-55)$
where u_s is the unit-step function.
Two cases of F_k^y will be studied:

$$(F_k^y)^1 = \begin{bmatrix} 2 & 0 \\ 0.6 & 0 \\ 0.2 & 0 \end{bmatrix} \text{ and } (F_k^y)^2 = \begin{bmatrix} 0 & 1.4 \\ 0 & 0.3 \\ 0 & 1.6 \end{bmatrix}.$$

The simulation results are illustrated by these figures :
Figure 1 and Figure 5 presents the actual fault vector$[f_{1,k},\ f_{2,k}]$ and their estimated values obtained by the proposed filter. The estimation of three components of state are presented in Figure2 and Figure6.
Convergence of the trace of the state covariance matrix $P^x_{k+1/k}$ are shown in Figure 3 and Figure7 respectively. Convergence of the trace of the fault covariance matrix $P^f_{k+1/k}$ are shown in Figure4 and Figure8 respectively. In table 2, the

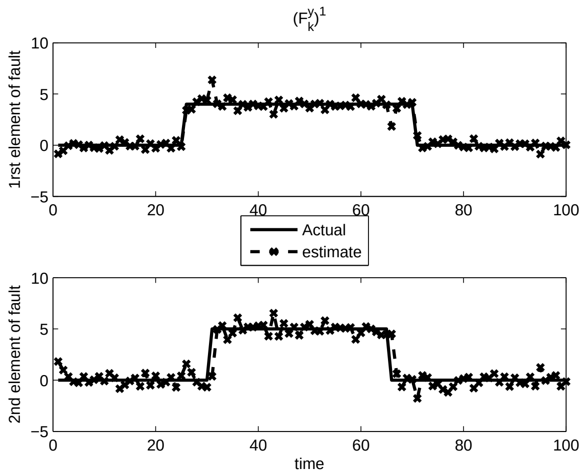

Figure. 1: Actual and estimated value of fault

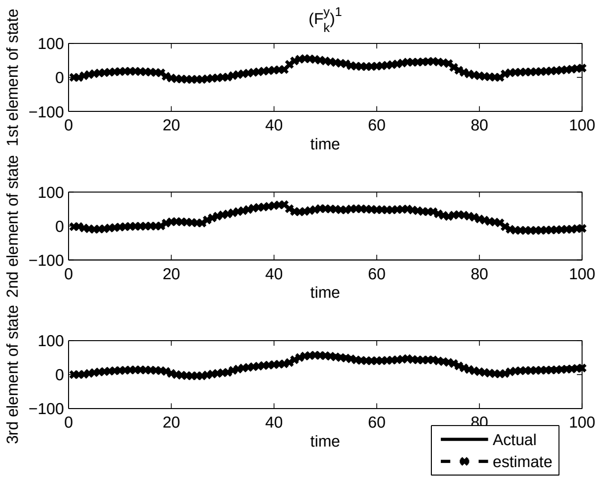

Figure. 2: Actual and estimated value of state

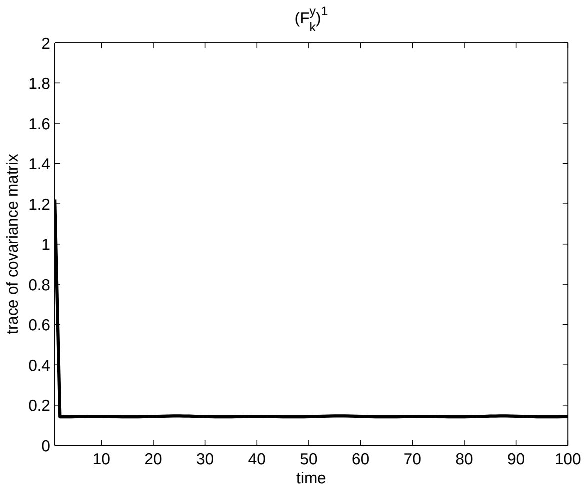

Figure. 3: Trace of the covariance matrix P_k^x

Table 2: Evaluation of the $rmse$ values

F_k^y	Filter	x_k^1	x_k^2	x_k^3	f_k^1	f_k^2
$(F_k^y)^1$	$Proposed\ filter$	0.1899	0.0863	0.0863	0.6631	0.1231
	$SFEF\ filter$	0.1899	0.0863	0.0863	1.128	0.1231
$(F_k^y)^2$	$Proposed\ filter$	0.2813	0.0986	0.0986	0.4757	0.8756
	$SFEF\ filter$	0.2813	0.0986	0.0986	0.9784	2.236

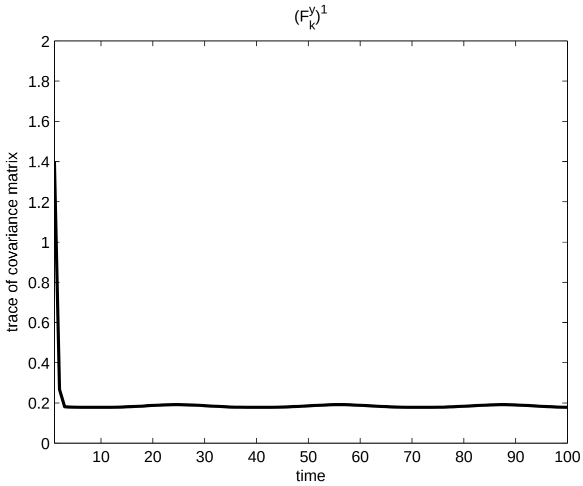

Figure. 4: Trace of the covariance matrix P_k^f

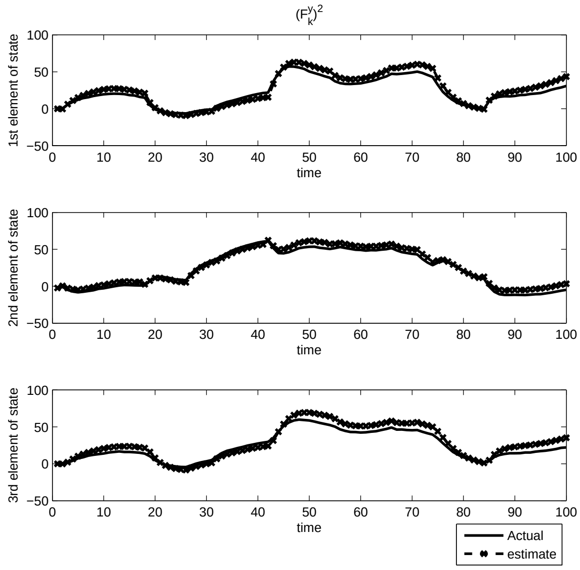

Figure. 6: Actual and estimated value of state

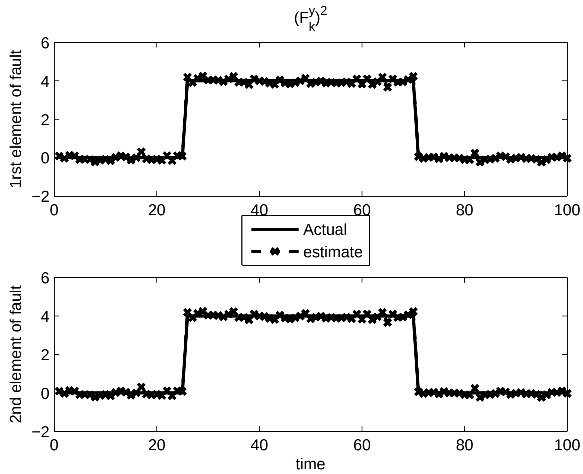

Figure. 5: Actual and estimated value of fault

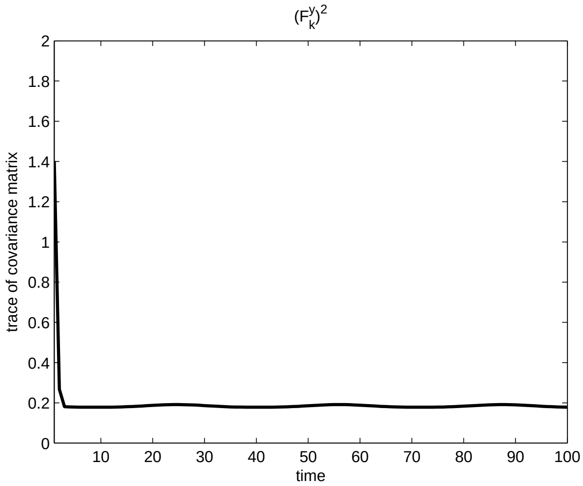

Figure. 7: Trace of the covariance matrix P_k^x

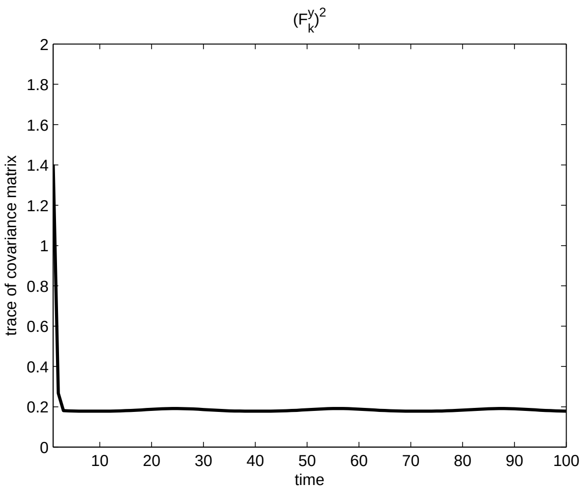

Figure. 8: Trace of the covariance matrix P_k^f

root square errors (RMSE) of the state $x_k = \begin{bmatrix} x_k^1 & x_k^2 & x_k^3 \end{bmatrix}^T$ and the fault $f_k = \begin{bmatrix} f_k^1 & f_k^2 \end{bmatrix}^T$ are given.
For example the RMSE of the first component of sate vector is calculated by:

$$RMSE\left(\tilde{x}_{1,k}\right) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(x_{1,k} - \hat{x}_{1,k}\right)^2}$$

According to the simulation results Figure1- Figure 8 and Tables 1 and 2, we may conclude with the following results : In all cases, the (State and fault estimation of linear stochastic discrete time systems) SFEF filter [17] and the proposed filter gives the same values of the RMSE of the state estimator error. The proposed filter gives smaller RMSE values of the fault estimation errors

V. Conclusions

This paper presented a recursive optimal filter for simultaneously estimating the state and fault in the presence of the unknown input in an unbiased minimum-variance sense for discrete-time linear stochastic system in presence of unknown disturbances, without any restriction on the direct feedthrough matrix of the system. The advantages of this filter are especially important in the case when we do not have any prior information about the fault or unknown input. This filter is applied efficiently to solve two problems. Firstly, it estimates the actuator and sensor faults simultaneously. Second, it establishes a comparative study with the existing literature results.

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